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*Publication date:*
1977

*Document Version*
Publisher's PDF, also known as Version of record

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*Citation (APA):*
Juul Rasmussen, J. (1977). *Nonlinear electron plasma wave in a cylindrical waveguide*. Risø National Laboratory. Risø-M No. 1911

Risø

Risø-M-1911

Risø-M-1911

Title and author(s)

Nonlinear electron plasma waves in a cylindrical waveguide.

by

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34 pages + tables + 6 illustrations

Date March 1977

Department or group

Physics

Group's own registration number(s)

Abstract

The nonlinear behaviour of the electron plasma wave propagating in a cylindrical plasma waveguide immersed in an infinite axial magnetic field is investigated using the Krylov-Bogoliubov-Mitropolsky perturbation method, by means of which the nonlinear Schrödinger equation, governing the long-term slow modulation of the wave amplitude, is deduced. From this equation the amplitude-dependent frequency and wave-number shifts are calculated, and it is found that the electron waves with short wavelengths are modulationally unstable with respect to long wavelength, low frequency perturbations. The growth rate of the instability is calculated and possible applications to experiments are discussed.

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F. 25-204

ISBN 87 550 0446-6

## CONTENTS

## 1. Introduction

Studies of the nonlinear propagation of finite amplitude waves have been of increasing importance in plasma physics during recent years. The amplitude-dependent changes of the frequency, the wavenumber and the amplitude of a monochromatic plane wave caused by weak nonlinearities can be investigated by means of the Krylov-Bogoliubov-Mitropolsky perturbation method[1]. This method was used in the pioneering works of Montgomery and Tidman[2] and Tidman and Stainer[3] to derive the nonlinear wavenumber and frequency shifts for unbounded electron plasma waves. However, a simplifying assumption (sufficient but not necessary) made by these authors prevented them from obtaining the long-term slow modulation of the wave amplitude, as pointed out by Kakutani and Sugimoto[4]. By using an extension of the Krylov-Bogoliubov-Mitropolsky perturbation procedure, Kakutani and Sugimoto[4] deduced a nonlinear Schrödinger equation govering the modulation of the amplitude of a monochromatic plane wave. The essence of their method lies in the systematic annihilation of all the secular terms arising in the perturbation expansion. They applied the method to unbounded electron plasma waves, magneto-acoustic waves, and ion acoustic waves. Later, the method was applied by Chan and Seshadri[5] to investigate the slow modulation of the ion plasma wave, taking into account finite ion temperature and non-vanishing electron inertia, contrary to the investigation in ref. 4. Further, the method has recently been applied to plane electron waves in a plasma stream[6] and finite amplitude shear Alfvén waves[7].

In the work reported here we used the method of Kakutani and Sugimoto to examine the nonlinear axial behaviour of the lowest-order electron plasma mode propagating in a cold plasma filling a cylindrical waveguide immersed in an essentially infinite axial magnetic field. This mode has a dispersion relation[8], $(\omega/k_{||})^2 = \omega_p^2/(k_{||}^2 + k_{\perp}^2)$, similar to that of ion acoustic waves in a plasma consisting of cold ions and isothermal electrons[9], with $1/k_{\perp}$ replacing the Debye length. Here $k_{||}$ and $k_{\perp}$ are the axial and perpendicular wavenumbers, respectively, and $\omega_p$ is the electron plasma frequency. Kakutani and Sugimoto[4] found, in their treatment of the ion acoustic wave, that short waves with wavenumbers larger than some critical wavenumber are modulationally unstable, while long waves are stable. Because of the similarity between the linear dispersion relations of the problem treated here and those of the ion acoustic waves, we could expect a similar result. Indeed, Manheimer[10] predicted theoretically that a long wavelength ($k_\perp >> k_{||}$), nonlinear electron plasma wave propagating in a cylindrical waveguide would steepen into a sharp density discontinuity, just like a nonlinear ion acoustic wave. This steepening of bounded electron plasma waves has been observed experimentally[11,12]. A nonlinear theory for electron waves in a plasma wave guide in a strong magnetic field was also studied by Jensen[13] in connection with plasma wave echo.

We find here that such electron plasma waves with short wavelengths in a cold plasma with stationary ions (infinite ion mass) are modulationally unstable; that is they are unstable for wavenumbers larger than the critical wavenumber $k_c = 5.29/r_o$, where $r_o$ is the radius of the waveguide. In the case of a temperate plasma of infinite extent, on the other hand, Kakutani and Sugimoto[4] found that the electron plasma waves were stable for all wavelengths, assuming stationary ions. It appears, however, that the inclusion of mobile ions in this case alters this conclusion; that is the electron plasma waves can become modulationally unstable with respect to long wavelength, low frequency perturbations[14].

The disposition of the present report is as follows. First, we review the linear theory of slow electron waves, i.e. $\omega/k << c$ (velocity of ligth), in a plasma-filled waveguide, and derive the linear dispersion relation for these modes in Sec. II. In Sec. III, we then deduce the nonlinear equations describing the axial behaviour of the lowest-order mode by neglecting the coupling to higher-order radial and azimuthal modes. In Sec. IV we carry out the perturbation expansion of the nonlinear equations, following Kakutani and Sugimoto[4], and obtain the first- and second-order solutions. The derivation of the nonlinear Schrödinger equation, describing the long-term slow modulation of the wave amplitude, is presented in Sec. V. The consequences of this nonlinear Schrödinger equation are considered in Sec. VI, where the amplitude-dependent frequency and wavenumber shifts are calculated. Further, the criteria for and growth rate of the modulational instability are deduced. This section also gives some equilibrium solutions to the nonlinear Schrödinger equation. Finally, in Sec. VII, the results are discussed and in conclusion we present a numerical example with reference to a Q-machine plasma.

## II. The Basic Equations and the Linear Dispersion Relation

Let us consider a perfectly conducting cylinder of radius $r_o$ filled with a cold plasma in an infinite axial magnetic field, as illustrated in Fig. 1. The electrons are constrained by the magnetic field to move only in the x-direction (field direction), and the ions will be assumed to form a stationary neutralizing background; we are only concerned with oscillations with frequencies much higher than the ion plasma frequency. Thus the equations describing the system are the fluid equations describing the conservation of electron number and momentum, and the Maxwell equations,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n u) = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{e}{m} E_x = 0 \qquad (2)$$

$$\nabla \cdot \underline{E} = \frac{e}{\varepsilon_o}(n_o \eta(r) - n) \qquad (3)$$

$$(\nabla(\nabla \cdot \underline{E}))_x - (\nabla^2 \underline{E})_x = \frac{1}{c^2}\left(\frac{\partial^2 E_x}{\partial t^2} - \frac{e}{\varepsilon_o}\frac{\partial}{\partial t}(n u)\right), \qquad (4)$$

where $n_o$ is the unperturbed electron density at the centre of the plasma (= ion density), $\eta(r)$ is a dimensionless function describing the radial density profile, n is the electron density, u is the fluid velocity in the x-direction and $\underline{E}$ is the electric field. In Eqs. (1)-(4) we made use of the fact that only the x-component $E_x$ of the electric field is affected by the presence of the plasma.[8] Substituting Eqs. (3) into (4) and considering only slow waves, i.e. $\omega/k \ll c$, we obtain the set of govering equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{e}{m}E_x = 0 \tag{6}$$

$$\frac{e}{\varepsilon_0}\frac{\partial n}{\partial x} + \frac{\partial^2 E_x}{\partial x^2} + \nabla_T^2 E_x = 0\,, \tag{7}$$

where $\nabla_T^2$ is the transverse Laplacian, here in cylindrical coordinates.

We first analyze the linear solutions of Eqs. (5)-(7). Thus we linearize the equations and obtain:

$$\frac{\partial n_1}{\partial t} + n_0\eta(r)\frac{\partial u_1}{\partial x} = 0 \tag{8}$$

$$\frac{\partial u_1}{\partial t} + \frac{e}{m}E = 0 \tag{9}$$

$$\frac{e}{\varepsilon_0}\frac{\partial n_1}{\partial x} + \frac{\partial^2 E}{\partial x^2} + \nabla_T^2 E = 0\,, \tag{10}$$

where $n_0\eta(r) + n_1 = n$. Any wave in the system must have a radial structure which in some way is determined by the radius of the waveguide, because E must vanish at the wall of the perfectly conducting waveguide. Equations (8)-(10) can be solved by separating the variables[8], thus we assume a wave dependence of the form:

$$E(r,\theta,x,t) \propto E_T(r,\theta)\exp[i(kx-\omega t)]\,. \tag{11}$$

Substitution of Eq. (11) and the similar forms for $n_1$ and $u_1$ into Eqs. (8)-(10), we find simply:

$$\nabla_T^2 E_T(r,\theta) + k^2\left(\frac{\omega_p^2}{\omega^2}\eta(r) - 1\right)E_T(r,\theta) = 0, \tag{12}$$

where $\omega_p^2 = n_0 e^2/\varepsilon_0 m$.

Equation (12) has the form of an eigenvalue equation with the boundary condition that $E_T(r=r_o,\theta) = 0$. The eigenfunctions can in principle be determined from $\eta(r)$. Here, for simplicity, we assume the unperturbed density to be constant across the waveguide, i.e. $\eta(r) \equiv 1$. The solution of Eq. (12) is then given in terms of Bessel functions:

$$E_T(r,\theta) = J_n(k_\perp r)\exp(-in\theta). \tag{13}$$

$J_n$ is the n'th order Bessel function of first kind, and the eigenvalues $k_\perp$ must satisfy the relation:

$$-k_\perp^2 + k^2\left(\frac{\omega_p^2}{\omega^2} - 1\right) = 0 \qquad \text{or}$$

$$\omega^2 = \omega_p^2 \frac{k^2}{k_\perp^2 + k^2}, \tag{14}$$

which is the dispersion relation for the axial modes. $k_\perp$ is determined by using the boundary condition $J_n(k_\perp \cdot r_o) = 0$:

$$k_\perp = \frac{p_{nm}}{r_o},$$

where $p_{nm}$ is the m'th zero of the n'th order Bessel function. The dispersion relation Eq. 14 now reads:

$$\omega^2 = \omega_p^2 \frac{(kr_o/p_{nm})^2}{1 + (kr_o/p_{nm})^2}. \tag{15}$$

Equation (15) is plotted in Fig. 2 for the lowest-order modes (n=0, m=1 and n=1, m=1 and n=o, m=2).

For the radial eigenfunction, we introduce the function $R_{nm}$ defined by:

$$R_{nm}(r) = J_n\left(\frac{\text{[illegible]}}{r_0} r\right). \tag{16}$$

With the above definition of m, m-1 will describe the number of nodes in the radial eigenfunction.

## III. Nonlinear Equations for the Axial Behaviour

In the nonlinear treatment of Eqs. (5)-(7) it is clear that the nonlinear terms $\partial(nu)/\partial x$ and $u(\partial u/\partial x)$ will couple the two azimuthal modes n and n' to produce density fluctuations at $n \pm n'$, as well as that the two axial modes k and k' will give fluctuations at $k \pm k'$; however, the coupling of radial modes is more complicated. Two radial modes m and m' will give rise to a driving term of the form $R_{nm}(r)\cdot R_{n'm'}(r)$, which can be expanded into terms of $R_{n\pm n'q}$ by using the fact that $R_{nm}$ forms an orthogonal set of eigenfunctions (Bessel functions Eq. (16)):

$$R_{nm}(r)\, R_{n'm'}(r) = \sum_q a_{qmm'}\, R_{n\pm n'q}(r) , \tag{17}$$

where

$$a_{qmm'} = \frac{\int_0^{r_0} r\, R_{nm}(r) R_{n'm'}(r)\, R_{n\pm n'q}(r)\, dr}{\int_0^{r_0} r\, R^2_{n\pm n'q}(r)\, dr} . \tag{18}$$

The coupling between the three modes $\omega$ k n m, $\omega$'k'n'm' and $\omega$"k"n"m" is resonant if the following conservation laws are satisfied[15-16]

$$\omega = \omega' + \omega'' \;, \quad k = k' + k'' \;, \quad n = n' + n'' \;, \tag{19}$$

which can be thought of as the conservation of wave energy, wave axial momentum and wave azimuthal momentum, respectively. The nonlinear coupling between the different azimuthal and radial modes has been considered both theoretically and experimentally by Laval et al.[15] and by Franklin et al.[16]. It should be noted that Franklin et al.[16] find that the decay of the (n=o, m=1) mode into the (n,m) mode is only possible for wavenumbers k larger than $k_{dnm}$ given by:

$$k_{dnm}^2 = 4(p_{nm}^2 - p_{01}^2)/3r_0^2. \tag{20}$$

Here we are only interested in the nonlinear axial behaviour of the (n=o, m=1) mode. This implies that all the harmonics generated by the nonlinear terms will have n=o. However, resonant decay may produce any two modes that satisfy the conditions (19). Neglecting parametric decay, we assume that the radial structure of the quantities is at all times discribed by the linear radial eigenfunction $R_{01}$. Accordingly, following Manheimer[10], we take,

$$\begin{aligned} E &= E(x,t)\, R_{01}(r) \\ n &= n_0 + n(x,t)\, R_{01}(r) \\ u &= u(x,t)\, R_{01}(r). \end{aligned} \tag{21}$$

Using Eqs. (21) we write Eqs. (5)-(7)

$$\frac{\partial n}{\partial t} R_{01} + n_0 \frac{\partial u}{\partial x} R_{01} = -\frac{\partial}{\partial x}(nu) R_{01}^2 = -\frac{\partial}{\partial x}(nu) \sum_q \alpha_{q11} R_{0q} \tag{22}$$

$$\frac{\partial u}{\partial t} R_{01} + \frac{e}{m} E R_{01} = -u \frac{\partial u}{\partial x} R_{01}^2 = -u \frac{\partial u}{\partial x} \sum_q \alpha_{q11} R_{0q} \tag{23}$$

$$\frac{e}{\epsilon_0} \frac{\partial n}{\partial x} R_{01} + \frac{\partial^2 E}{\partial x^2} R_{01} + E\, \nabla_T^2 R_{01} = 0. \tag{24}$$

Where $\alpha_{q11}$ is given by Eq. (18), it is clear that $\alpha_{111}$ is larger than any other $\alpha_{q11}$; that is, the coupling between different axial modes of the same radial structure is stronger than the coupling to other radial modes. If we now only retain the $\alpha_{111}$ driving terms - that is, we neglect coupling to other radial modes - we get:

$$\frac{\partial n}{\partial t} + n_o \frac{\partial u}{\partial x} + \alpha \frac{\partial (nu)}{\partial x} = 0 \tag{25}$$

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{e}{m} E = 0 \tag{26}$$

$$\frac{e}{\varepsilon_o} \frac{\partial n}{\partial x} + \frac{\partial^2 E}{\partial x^2} - \frac{1}{\beta^2} E = 0 , \tag{27}$$

where $\beta = r_o / p_{o1} \quad ( p_{o1} = 2.405 )$ ,

$$\alpha = \int_0^{p_{o1}} z J_o^3(z)\, dz \Big/ \int_0^{p_{o1}} z J_o^2(z)\, dz \simeq 0.72 . \tag{28}$$

Normalizing the quantities n with respect to $n_o$, u with respect to the characteristic velocity $u_o = \omega_p \cdot \beta$, and E with respect to the characteristic electric field $E_o = m\beta\omega_p^2/e$, and taking $\beta$ and $\omega_p^{-1}$ as characteristic length and time, respectively, we get these governing equations for the nonlinear axial behaviour of the (n=o, m=1) mode:

$$\frac{\partial n}{\partial t} + \frac{\partial u}{\partial x} + \alpha \frac{\partial (nu)}{\partial x} = 0 \tag{29}$$

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + E = 0 \tag{30}$$

$$\frac{\partial n}{\partial x} + \frac{\partial^2 E}{\partial x^2} - E = 0 \tag{31}$$

And the linear dispersion relation Eq. (15) takes the form

$$k^2 - \omega^2 - \omega^2 k^2 = 0 \tag{32}$$

These equations (29-32) are the basic equations for the perturbation expansion carried out in the following sections.

## IV. The Perturbation Expansion and the Second-Order Solution

In order to examine the nonlinear behaviour of wave solutions to Eqs. (29)-(31), with small but finite amplitude over time-scales that are large compared to the wave period and spatial scales that are large compared to the wavelength, we use the method of Kakutani and Sugimoto[4] based on the Krylov-Bogoliubov-Mitropolsky perturbation method[1]. The essence of this method lies in the systematic annihilation of all the secular terms arising in the perturbation expansion. This annihilation then gives the long-term and large-scale behaviour of the wave amplitude. The term "small amplitude" should here be understood in the sense that the electron movement in the wave electric field is much smaller than $\beta$. This implies that the amplitude of the wave electric field is much smaller than $E_o = m\beta\omega_p^2/e$.

We introduce a "smallness parameter" $\varepsilon$ and expand all quantities in Eqs. (29)-(31) around the unperturbed uniform state in powers of $\varepsilon$ in the following form[4]

$$\begin{Bmatrix} E \\ u \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \varepsilon \begin{Bmatrix} E_1 \\ u_1 \\ n_1 \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} E_2 \\ u_2 \\ n_2 \end{Bmatrix} + \varepsilon^3 \begin{Bmatrix} E_3 \\ u_3 \\ n_3 \end{Bmatrix} + \cdots\cdots\cdots , \qquad (33)$$

where $\varepsilon$ indicates the relative "smallness" of the terms on the right-hand side, and it will later be set equal to unity. We then choose a monochromatic plane wave as a starting solution to $E_1$:

$$E_1 = a\exp(i\psi) + \bar{a}\exp(-i\psi), \qquad (34)$$

where a is the complex amplitude of the perturbed electric field normalized with respect to $E_o$, i.e. $|a| << 1$, $\bar{a}$ denotes the complex conjugated to a, and $\psi$ is the phase defined as

$\psi = kx - \omega t$, where the wavenumber k and the frequency $\omega$ must satisfy the linear dispersion relation (Eq. (32)). We assume that each coefficient of the $\varepsilon$ power depends on x and t through a, $\bar{a}$ and $\psi$. The complex amplitude a is further assumed to be a slowly varying function of x and t described through the relations:

$$\frac{\partial a}{\partial t} = \varepsilon A_1(a,\bar{a}) + \varepsilon^2 A_2(a,\bar{a}) + \cdots\cdots \tag{35}$$

$$\frac{\partial a}{\partial x} = \varepsilon B_1(a,\bar{a}) + \varepsilon^2 B_2(a,\bar{a}) + \cdots\cdots \tag{36}$$

together with the complex conjugate relations, while the phase $\psi$ remains unchanged from the linear limit, i.e. $\psi = kx - \omega t$, because nonlinear effects on the phase will be taken into account through the "phase part" of the complex amplitude. The unknown functions $A_1, B_1, A_2, B_2 \ldots$ will be determined by eliminating the secular terms in the perturbation expansion.

Substituting the expansions Eq. (33) into Eqs. (29)-(31) and collecting terms with the same powers in $\varepsilon$ we obtain:

$$\begin{aligned}
&\varepsilon\left\{-\omega\frac{\partial n_1}{\partial\psi} + k\frac{\partial u_1}{\partial\psi}\right\} + \varepsilon^2\left\{-\omega\frac{\partial n_2}{\partial\psi} + k\frac{\partial u_2}{\partial\psi} + \alpha k\frac{\partial(u_1 n_1)}{\partial\psi} + \left(A_1\frac{\partial n_1}{\partial a} +\right.\right.\\
&\left.\left.B_1\frac{\partial u_1}{\partial a} + c.c.\right)\right\} + \varepsilon^3\left\{-\omega\frac{\partial n_3}{\partial\psi} + k\frac{\partial u_3}{\partial\psi} + \alpha k\frac{\partial(n_1 u_2)}{\partial\psi} + \alpha k\frac{\partial(n_2 u_1)}{\partial\psi} + \left(A_2\frac{\partial n_1}{\partial a}\right.\right.\\
&\left.\left.+A_1\frac{\partial n_2}{\partial a} + B_2\frac{\partial u_1}{\partial a} + B_1\frac{\partial u_2}{\partial a} + \alpha B_1\frac{\partial(n_1 u_1)}{\partial a} + c.c.\right)\right\} + O(\varepsilon^4) = 0
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\varepsilon\left\{-\omega\frac{\partial u_1}{\partial\psi} + E_1\right\} + \varepsilon^2\left\{-\omega\frac{\partial u_2}{\partial\psi} + E_2 + \alpha k u_1\frac{\partial u_1}{\partial\psi} + \left(A_1\frac{\partial u_1}{\partial a} + c.c.\right)\right\} + \varepsilon^3\left\{-\omega\frac{\partial u_3}{\partial\psi}\right.\\
&\left.+E_3 + \alpha k u_1\frac{\partial u_2}{\partial\psi} + \alpha k u_2\frac{\partial u_1}{\partial\psi} + \left(A_2\frac{\partial u_1}{\partial a} + A_1\frac{\partial u_2}{\partial a} + \alpha B_1 u_1\frac{\partial u_1}{\partial a} + c.c.\right)\right\} + O(\varepsilon^4) = 0
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\varepsilon\left\{k\frac{\partial n_1}{\partial\psi} + k^2\frac{\partial^2 E_1}{\partial\psi^2} - E_1\right\} + \varepsilon^2\left\{k\frac{\partial n_2}{\partial\psi} + k^2\frac{\partial^2 E_2}{\partial\psi^2} - E_2 + \left(B_1\frac{\partial n_1}{\partial a} + 2kB_1\frac{\partial^2 E_1}{\partial a\,\partial\psi} +\right.\right.\\
&\left.\left.c.c.\right)\right\} + \varepsilon^3\left\{k\frac{\partial n_3}{\partial\psi} + k^2\frac{\partial^2 E_3}{\partial\psi^2} - E_3 + \left[B_2\frac{\partial n_1}{\partial a} + B_1\frac{\partial n_2}{\partial a} + 2kB_2\frac{\partial^2 E_1}{\partial a\,\partial\psi} + B_1^2\frac{\partial^2 E_1}{\partial a^2} +\right.\right.\\
&\left.\left.B_1\bar{B}_1\frac{\partial^2 E_1}{\partial a\,\partial\bar{a}} + \left(B_1\frac{\partial B_1}{\partial a} + \bar{B}_1\frac{\partial B_1}{\partial\bar{a}}\right)\frac{\partial E_1}{\partial a} + 2kB_1\frac{\partial^2 E_2}{\partial\psi\,\partial a} + c.c.\right]\right\} + O(\varepsilon^4) = 0,
\end{aligned} \tag{39}$$

where c.c. stands for the complex conjugate. From the set of equations for $o(\varepsilon)$ and Eq. (34), we get the following starting solutions for $n_1$ and $u_1$ that satisfy the linear dispersion relation Eq. (32):

$$n_1 = -\frac{ik}{\omega^3}\{a\exp(i\psi) - \bar{a}\exp(-i\psi)\} \tag{40}$$

$$u_1 = -\frac{i}{\omega}\{a\exp(i\psi) - \bar{a}\exp(-i\psi)\} \tag{41}$$

We now seek the set of second-order solutions. Substituting the first-order solutions (Eqs. (34)(40) and (41)) into the set of equations for $o(\varepsilon^2)$, we arrive at the differential equations for $n_2, u_2$ and $E_2$:

$$-\omega\frac{\partial n_2}{\partial\psi} + k\frac{\partial u_2}{\partial\psi} = \frac{i}{\omega}(A_1\frac{k}{\omega} + B_1)\exp(i\psi) + i\alpha 2\frac{k^2}{\omega^3}a^2\exp(2i\psi) + c.c. \tag{42}$$

$$-\omega\frac{\partial u_2}{\partial\psi} + E_2 = \frac{i}{\omega}A_1\exp(i\psi) + i\alpha\frac{k}{\omega^3}a^2\exp(2i\psi) + c.c. \tag{43}$$

$$k\frac{\partial n_2}{\partial\psi} + k^2\frac{\partial^2 E_2}{\partial\psi^2} - E_2 = ikB_1(\frac{1}{\omega^2} - 2)\exp(i\psi) + c.c. \tag{44}$$

By appropriate manipulation of Eqs. (42)-(44) and using the dispersion relation Eq. (32), we deduce the differential equation satisfied by $E_2$:

$$\frac{\partial^2 E_2}{\partial\psi^2} + E_2 = \frac{2i}{\omega^3}(A_1 + \frac{\omega^3}{k^3}B_1)\exp(i\psi) + i\alpha 3\frac{k}{\omega^4}a^2\exp(2i\psi) + c.c. \tag{45}$$

In order that $E_2$ should be free of secular terms, we must require that the coefficient of exp $(\pm i\psi)$, i.e. the resonant term with respect to the solutions of the homogeneous part of (45), in Eq. (45) vanishes. This yields

$$A_1 + v_g B_1 = 0 , \tag{46}$$

and its complex conjugate, where $v_g = \omega^3/k^3$ is the group velocity of the wave, $v_g = d\omega/dk$, which is seen from the dispersion relation Eq. (32). By virtue of the relations (35) and (36), we can regard $A_1$ and $B_1$, respectively, as $\partial a/\partial t_1$ and $\partial a/\partial x_1$ to the lowest order in $\varepsilon$ where $t_1 = \varepsilon t$ and $x_1 = \varepsilon x$. Thus we may interpret Eq. (46) as:

$$\frac{\partial a}{\partial t_1} + v_g \frac{\partial a}{\partial x_1} = 0, \tag{47}$$

and its complex conjugate, which shows that, to the lowest order in $\varepsilon$, the amplitude a is constant in a frame of reference moving with the group velocity, i.e. $a$ depends on $t_1$ and $x_1$ only through $\xi = [x_1 - v_g t_1 = \varepsilon(x - v_g t)]$.

By using Eqs. (46) and (32) we can now evaluate the secular free second-order solutions from Eqs. (45), (42) and (43) and obtain

$$E_2 = -\frac{ik}{\omega^4}\alpha a^2 \exp(2i\psi) + b\exp(i\psi) + c.c. \tag{48}$$

$$u_2 = -\frac{k(1+\omega^2)}{2\omega^5}\alpha a^2 \exp(2i\psi) + \left(\frac{\omega}{k^3}B_1 - \frac{i}{\omega}b\right)\exp(i\psi) + c.c. + c_1 \tag{49}$$

$$n_2 = -\frac{k^2(1+3\omega^2)}{2\omega^6}\alpha a^2 \exp(2i\psi) + \left(\frac{1-k^2}{k^2}B_1 - \frac{ik}{\omega^2}b\right)\exp(i\psi) + c.c. + c_2, \tag{50}$$

where the complex functions b and $\bar{b}$, and the real functions $c_1$ and $c_2$, are constants with respect to $\psi$ and depend on $a$ and $\bar{a}$ alone. They should be determined from the non-secular conditions of higher orders.

## V. Third-Order Equations and the Nonlinear Schrödinger Equation.

We now proceed to the set of third-order equations. Substituting the first-order (Eqs. (34), (40) and (41)) and the second-order solutions (Eqs. (48)-(50)) into the set of equations for $o(\varepsilon^3)$, we can derive the following equations for $n_3$, $u_3$ and $E_3$:

$$-\omega \frac{\partial n_3}{\partial \psi} + k \frac{\partial u_3}{\partial \psi} = D_{13} \exp(3i\psi) + D_{12} \exp(2i\psi) + D_{11} \exp(i\psi) + D_{10} + c.c. \tag{51}$$

$$-\omega \frac{\partial u_3}{\partial \psi} + E_3 = D_{23} \exp(3i\psi) + D_{22} \exp(2i\psi) + D_{21} \exp(i\psi) + D_{20} + c.c. \tag{52}$$

$$k \frac{\partial n_3}{\partial \psi} + k^2 \frac{\partial^2 E_3}{\partial \psi^2} - E_3 = D_{32} \exp(2i\psi) + D_{31} \exp(i\psi) + D_{30} + c.c., \tag{53}$$

where the terms $D_{ij}$ are relegated to the appendix in view of their complexity. The secular-free conditions for the third--order solutions consist of two parts: One is the annihilation of the secularity-producing constants, and the other is that of secularity-producing resonant terms. From the former, we obtain a sufficient number of equations to determine $c_1$ and $c_2$ by averaging Eqs. (51)-(53) with respect to the fast phase $\psi$:

$$D_{10} = 0 \quad , \quad D_{20} = -D_{30} \quad ,$$

and

$$C_1 = \frac{1}{\omega^2} \frac{2\, k/\omega + V_g}{V_g^2 - 1} \alpha a \bar{a} + d_1 \tag{54}$$

$$C_2 = \frac{1}{\omega^2} \frac{2\, \omega^2/k^2 + 1}{V_g^2 - 1} \alpha a \bar{a} + d_2 \quad , \tag{55}$$

where $d_1$ and $d_2$ are absolute constants with respect to $\psi$, $a$ and $\bar{a}$ (for a further discussion of the interpretation of $d_1$ and $d_2$, the appendix of Ref. 4 should be consulted).

By manipulating Eqs. (51)-(53) in the same way as when deriving Eq. (45), we can deduce the differential equation determining $E_3$. The requirement that $E_3$ be free of secularity arising from the resonant terms yields

$$\frac{1}{k^2} D_{31} + \frac{1}{\omega^2} D_{21} + \frac{1}{k\omega} D_{11} = 0 .$$

Inserting the expressions for $D_{31}$, $D_{21}$ and $D_{11}$ and using Eqs. (54) and (55), we obtain after simple but lengthy algebra the following differential equation for $a$ (as well as $\bar{a}$):

$$i\,(A_2 + v_g B_2) + P\,\left(B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}}\right) = Q\,|a|^2 a + Ra \qquad (56)$$

where

$$P = -\frac{3\omega^5}{2k^4} = \frac{1}{2}\frac{dv_g}{dk} \qquad (57)$$

$$Q = \omega^3 \frac{k^{10} - 15k^6 - 35k^4 - 30k^2 - 9}{4k^6(k^4 + 3k^2 + 3)}\,\alpha^2 \qquad (58)$$

$$R = (kd_1 + \frac{\omega}{2} d_2)\,\alpha \qquad (59)$$

With the aid of Eqs. (35) and (36) we note that $A_2$, $B_2$ and $B_1(\partial B_1/\partial a) + \bar{B}_1(\partial B_1/\partial \bar{a})$ can be interpreted, respectively, as $\partial a/\partial t_2 - A_1/\varepsilon$, $\partial a/\partial x_2 - B_1/\varepsilon$ and $\partial^2 a/\partial x_1^2$, where $t_2 = \varepsilon^2 t$, $x_2 = \varepsilon^2 x$ and $x_1 = \varepsilon x$. Equation (56) then reads:

$$i\left(\frac{\partial a}{\partial t_2} + v_g \frac{\partial a}{\partial x_2}\right) + P\frac{\partial^2 a}{\partial x_1^2} = Q\,|a|^2 a + Ra . \qquad (60)$$

It was shown that a only depends on $x_1$ and $t_1$ through $\xi = x_1 - v_g t_1$ (Eq. (47)), this leads to the introduction of the following coordinate transformation[4]:

$$\xi = \frac{1}{\varepsilon}(x_2 - v_g t_2) = x_1 - v_g t_1 = \varepsilon(x - v_g t)$$

$$\tau = t_2 = \varepsilon t_1 = \varepsilon^2 t \,, \tag{61}$$

by means of which we can convert (60) to a nonlinear Schrödinger equation:

$$i\frac{\partial a}{\partial \tau} + P\frac{\partial^2 a}{\partial \xi^2} = Q|a|^2 a + Ra \,. \tag{62}$$

The arbitrary constants $d_1$ and $d_2$ (in the expression for R, Eq. (59)) and therefore R may be determined if appropriate initial or boundary conditions are specified. However, the linear interaction term Ra only causes a simple phase shift, and can be removed from Eq. (62) by the substitution[17)]

$$a \rightarrow a \; \exp\left(-i\int^{\tau} R(\tau')\,d\tau'\right) . \tag{63}$$

It is interesting to note that the variables $\tau$ and $\xi$ introduced into Eq. (61) are identical to the stretched variables used by Tanuiti et al.[18-19)] in deriving the nonlinear Schrödinger equation for nonlinear wave modulation.

## VI. Nonlinear Dispersion Relation and Modulational Stability

Solutions to the nonlinear Schrödinger equation (62) have been extensively studied[4,18-27] in connection with nonlinear propagation of waves of various kinds. In this section we consider some of these solutions and their physical consequences.

We note that Eq. (62) is complex, therefore the solutions are also complex. Thus we introduce the real functions $\rho(\xi,\tau)$ and $\theta(\xi,\tau)$ representing the real and imaginary parts of a through:

$$a = \rho \exp(i\theta). \tag{64}$$

Substituting this expression into Eq. (62), we have

$$\frac{\partial\rho}{\partial\tau} + 2P\frac{\partial\rho}{\partial\xi}\frac{\partial\theta}{\partial\xi} + \rho P\frac{\partial^2\theta}{\partial\xi^2} = 0 \tag{65}$$

and

$$\frac{\partial\theta}{\partial\tau} - P\left(\frac{1}{\rho}\frac{\partial^2\rho}{\partial\xi^2} - \left(\frac{\partial\theta}{\partial\xi}\right)^2\right) + Q\rho^2 + R = 0. \tag{66}$$

It is easily seen that Eq. (65) can be written in the form of a conservation equation for $\rho^2$:

$$\frac{\partial(\rho^2)}{\partial\tau} + \frac{\partial}{\partial\xi}\left(\rho^2 2P\frac{\partial\theta}{\partial\xi}\right) = 0 \tag{67}$$

Noting that $\rho^2 = |a|^2$, Eq. (67) expresses that the "wave energy", proportional to $|a|^2$, is conserved. In order to obtain the nonlinear dispersion relation for the electron plasma wave, we consider the plane wave solution to Eq. (62):

$$a = \rho_0 \exp[i(\kappa\xi - \phi\tau)] \tag{68}$$

where $\rho_0$, $\kappa$ and $\phi$ are constants and $\rho_0$ the value for $|a|$ at infinity, i.e. in Eq. (64) $\rho=\rho_0$ and $\theta=\theta_0=\kappa\xi-\phi\tau$. Using Eqs. (65) and (66), we obtain the dispersion relation

$$\phi - P\kappa^2 = Q\rho_0^2 + R \ , \tag{69}$$

and the group velocity $U_g=2P\kappa$. Recalling that

$$E_1 = a \exp[i(kx-\omega t)] + c.c.,$$

we have

$$E_1 = \rho_0 \exp\{i[(k+\varepsilon\kappa)x - (\omega+\varepsilon\kappa v_g+\varepsilon^2\phi)t]\} + c.c.$$

From this we get the nonlinear dispersion relation

$$\omega(k+\Delta k,\rho_0) \simeq (\omega + v_g\Delta k + P(\Delta k)^2) + \varepsilon^2(Q\rho_0^2+R), \tag{70}$$

where $\Delta k=\varepsilon\kappa$.

The first bracket in (70) is the term obtained by Taylor expansion about k and results from linear dispersion, that is:

$$\omega(k+\Delta k) \simeq \omega + v_g\Delta k + \frac{1}{2}\frac{dv_g}{dk}(\Delta k)^2 .$$

The second term in Eq. (70) shows the effect of finite amplitude on the dispersion relation. Suppose that we now consider an initial value problem, in which we give the amplitude and the wave number as spatially constant, at say t=o. Then, with the discussion of the R-term in Sec. V in mind, we see from (70) that the amplitude-dependent frequency shift induced by the nonlinearities is given by (setting the expansion parameter equal to unity)

$$\Delta\omega = Q\rho_o^2 .$$

By means of Eq. (58) the relative frequency shift can be written

$$\frac{\Delta\omega}{\omega} = \omega^2 \frac{k^{10} - 15k^6 - 35k^4 - 30k^2 - 9}{4k^6(k^4 + 3k^2 + 3)} \alpha^2 \rho_o^2 . \tag{71}$$

Conversely, if we consider a boundary value problem (which is more applicable from an experimental point of view) in which the progressive wave is excited with a time-independent frequency and amplitude at some spatial boundary, e.g. x=o in a semi-infinite plasma, the nonlinearities will cause an amplitude-dependent wavenumber shift as given by (Eq. (70))

$$\Delta k = -\frac{Q}{v_g} \rho_o^2 .$$

And the relative wavenumber shift can be written

$$\frac{\Delta k}{k} = -\frac{k^{10} - 15k^6 - 35k^4 - 30k^2 - 9}{4k^4(k^4 + 3k^2 + 3)} \alpha^2 \rho_o^2 . \tag{72}$$

In Fig. 3 we have plotted $\Delta\omega/\omega$ and $\Delta k/k$ as function of k. These relative frequency and wave number shifts are small of second order in the normalized amplitude $\rho_o$, which is indicated by the $\epsilon^2$ factor in Eq. (70). However, from (72) we see that $|\Delta\omega/\omega|$ and $|\Delta k/k| \to \infty$ when $k \to o$ and that $|\Delta k/k| \to \infty$ for $k \to \infty$ that is the expansion Eq. (33) breaks down for both small k-values and large k-values, and the results obtained are only valid in the intermediate range.

In addition to the amplitude-dependent frequency and wavenumber shifts, the nonlinearities can in general cause a slow modulation, described by Eq. (62), of the amplitude of the electron wave. From the theory of the nonlinear Schrödinger equation (see e.g. Refs. 4, 18-27), we find that the plane electron wave is modulationally stable or unstable according to whether $PQ > o$ or $PQ < o$. Inspection of Eqs. (57) and (58) shows that $PQ \gtrless o$ according as $k \lessgtr k_c = 2.2$. This means that short waves, $k > k_c$, are modulationally unstable, while long waves, with $k < k_c$, are stable. In unnormalized quantities the critical wavenumber is given by

$$k_c = \frac{2.2}{\beta} = \frac{5.29}{r_o}, \tag{73}$$

($k_c$ is marked on Fig. 2).

In order to find the growth rate and wavenumber for the modulation, we consider a small perturbation of the stationary plane wave (68). Thus we write:

$$\rho = \rho_o + \delta\rho \, Re\{exp[i(K\xi - \Omega\tau)]\} \qquad \text{and}$$

$$\theta = \theta_o + \delta\theta \, Re\{exp[i(K\xi - \Omega\tau)]\},$$

that is $\delta\rho$ and $\delta\theta$ account for the modulation in amplitude and phase, respectively. Substitution into Eqs. (65)-(66) and linearizing give the dispersion relation between $\Omega$ and $K$

$$(\Omega - Ku_g)^2 = P^2[(K^2 + \frac{Q}{P}\rho_o^2)^2 - \frac{Q^2}{P^2}\rho_o^4] \tag{74}$$

where $U_g$ is defined in connection with Eq. (69). For PQ < o it is seen that $\Omega$ has a imaginary part for $K < (2\,|Q/P|)^{\frac{1}{2}}\,\rho_o$. Hence, a long wave disturbance of the electron wave will grow, as expected. From Eq. (74) the growth rate of the modulation is obtained as:

$$\gamma = Im(\Omega) = |P|\,K\,(2\frac{|Q|}{|P|}\rho_o^2 - K^2)^{1/2} \tag{75}$$

It has the maximum value of

$$\gamma_m = Q\,\rho_o^2 \tag{76}$$

at

$$K = K_m = (|Q/P|)^{1/2}\rho_o\ , \tag{77}$$

where Q and P are given by Eqs. (57)-(58). (Note that $\gamma_m$ and $K_m$ are independent of the R term). The dependence of $\gamma$ on K is shown in fig. 4, and the variation of $\gamma_m$ and $K_m$ with respect to k is shown in fig. 5. The modulational instability is seen to be a weak instability, $\gamma_m$ is of second order in the small amplitude. In the next section we discuss this further and give a numerical example.

The physical mechanism of the modulational instability can be understood in the following simple way. Suppose a small modulation perturbation is applied to the plane wave. If, for instance, we consider a boundary value problem, we see from Eq. (72), since Q > o, that the wavenumber of the plane wave is smaller in the crest of the modulation than in the trough. Since P < o, the group velocity then becomes greater in the crest than in the trough, resulting in a pile-up of wave energy in the crest of the modulation. This implies growth of the modulation perturbation.

For $PQ > o$, i.e. long electron waves with $k < k_c$, equation (74) shows that $\Omega$ has only real solutions and that the waves are modulationally stable, in accordance with the discussion above. In this case the nonlinear terms in (65)-(66) will generate harmonics and thus generally steepen the waveform. However, the "dispersion" term P will control the steepening and prevent a break-down of the waveform, since $|a|^2$ is conserved (Eq. (67)). Steepening of a long wavelength electron plasma wave in a plasma waveguide was predicted theoretically by Manheimer[10)], and has been investigated experimentally by Ikezi et al.[11)] and Saeki[12)] in Q-machine plasma. In Refs. 11 and 12 the nonlinear behaviour is described in terms of the Korteweg-de Vries equation. In this connection it is interesting to note that Johnson[28)] recently showed that the long wave limit of the nonlinear Schrödinger equation coincides with the short wave limit of the Korteweg-de Vries equation, in the case where these equations were applied to short and long water waves on shear flows, respectively.

Finally, in addition to the plane wave solution described above, the nonlinear Schrödinger equation has equilibrium solutions exhibiting the dynamical balance between nonlinear and dispersion effects of the form

$$a = \rho(\xi - U_g\tau)\exp[i(\kappa\xi - \phi\tau)] \,, \tag{78}$$

where $\kappa$, $\phi$ and $U_g$ are defined in connection with Eqs. (68) and (69). By substituting (78) into (65) and (66) we arrive at

$$\rho'' = \frac{Q}{P}\rho^3 - \frac{Q}{P}\rho_0^2\,\rho \,,$$

where the differentiation concerns $(\xi - U_g\,\tau)$. Integrating once gives

$$\frac{1}{2}(\rho')^2 + V(\rho) = F_0 \,, \tag{79}$$

with

$$V(\rho) = -\frac{1}{2}\frac{Q}{P}\left(\frac{1}{2}\rho^4 - \rho_o^2\rho^2\right),$$

where $F_o$ is an arbitrary constant. This equation is equivalent to the classical equation of motion for a unit mass with total energy $F_o$ under the potential $V(\rho)$, as noted by Kakutani and Sugimoto[9]. With this analogy in mind, it is easily found from the form of $V(\rho)$ that for $Q/P > o$ (i.e. modulational stability) bounded solutions only exist for $o < F_o\ (Q/4P)\rho_o^4$. These solutions are in general expressible in terms of Jacobi elliptic functions[29]. In the special case where $F_o = (Q/P)\rho_o^4$, $\rho(\xi - U_g\tau)$ represents the so-called "phase jump" or "envelope shock" expressed as:

$$\rho(\xi - u_g\tau) = \rho_o \tanh\left[-\sqrt{\frac{Q}{2P}}\rho_o\cdot(\xi - u_g\tau)\right]. \qquad (80)$$

This "envelope shock" is plotted in fig. 6a. It is seen to propagate in (x,t) space with approximately the group velocity $v_g$ of the electron wave (the correction $U_g$ is much smaller than $v_g$). For $Q/P < o$, on the other hand, two types of bounded solutions exist: One for $F_o > o$, "large" amplitude waves, and one for $-|Q/4P|\rho_o^4 < F_o < o$, "small" amplitude waves. Both may generally be expressed by Jacobi elliptic functions. In particular, $F_o = o$, we obtain the solitary wave of the form

$$\rho(\xi - u_g\tau) = \sqrt{2}\,\rho_o\, \mathrm{sech}\left[\sqrt{\frac{|Q|}{|P|}}\rho_o\cdot(\xi - u_g\tau)\right]. \qquad (81)$$

Also this solitary wave propagates in the (x,t) space approximately with the group velocity $v_g$. The width of the solitary wave, which is plotted in fig. 6b, is seen to agree with the wavelength of the unstable modulation mode with maximum growth rate (Eq.77). This leads to the conjecture that the modulation of the electron wave is eventually deformed into the solitary wave described by (81). Numerical solutions of the nonlinear Schrödinger equation support this conjecture[30]. Mention should also be made of the numerical calculations by Yajima and Outi[31]. They show that a solitary wave solution like (81) is so stable that it preserves its identity in spite of the nonlinear interactions. For this reason, the solitary wave is often called the "envelope soliton".

The deformation of the electric field into the envelope soliton is accompanied by formation of the density cavity as seen from the first-order solutions Eqs. (34), (40). This may also be understood in the following way. The ponderomotive force[32], arising from the gradients in the electric field amplitude, pushes away the electrons from regions with high field intensity, and thus digs the cavity in the density. Propagation of such density cavities enclosing an electric field "soliton" have been observed by Kim et al.[33] and Ikezi et al.[34] in experiments where the plasma could be treated as infinite.

## VII. Discussion and Conclusion

Calculations were presented of the nonlinear behaviour of a finite amplitude electron plasma wave propagating in a cold plasma filling a cylindrical waveguide immersed in an infinite axial magnetic field. By means of the Krylov-Bogoliubov-Mitropolsky perturbation method[1] extended by Kakutani and Sugimoto[4], we derived the nonlinear Schrödinger equation for the long-term slow modulation of the wave amplitude of the lowest order mode ($n=o$, $m=1$). From this nonlinear Schrödinger equation we calculated the amplitude-dependent shifts in frequency and wavenumber introduced by the nonlinearities. Further, we found that the electron plasma wave with short wavelength was modulationally unstable with respect to long wavelength low-frequency perturbations, contrary to the case for electron waves in infinite plasma with stationary ions[4]. We have thus an example of a nonlinear effect introduced by a finite geometry, which is often met in experiments.

The basic assumption we made in setting up the nonlinear governing equations in Sec. III was that the transverse structure of the wave field was described by the linear transverse eigenfunctions even in the nonlinear limit. In other words, we assumed that the nonlinear axial propagation of one mode would not alter its transverse structure. This assumption is justified by the fact that, at sufficiently strong magnetic fields, the electrons will only move in the field direction, and the transverse structure of the wavefield will not be affected by the presence of the plasma at all, as also noted in Sec. II. However, the coupling to other transverse modes, which was neglected, is, of course still possible, as discussed in Sec. III. Further, the critical wave number $k_c = 5.29/r_o$ (Eq. (73)) for modulational instability is larger than the low wave number limit $k_{d11} = 3.45/r_o$ (Eq. (20)) for resonant decay to the ($n = 1$, $m = 1$) mode (see fig. 2). Thus the modulational instability of the ($n=o$, $m=1$) mode will evolve in competition with the decay of this mode to the higher order transverse modes (for $k_c < k < k_{d02}$, only decay to the ($n = 1$, $m = 1$) mode is possible). In spite of the possible decay of the ($n = o$, $m = 1$) mode, which may extract energy from this wave, the theory considered here will still be valid, but the other nonlinear processes not taken into account may limit the growth of the modulational instability. It should be noted that the theory can be applied to the higher-order modes too, by changing $\alpha$ and $\beta$ appropriately.

In a real plasma there will always be a finite temperature, but our calculations can still be applied in the case where $r_o \gg \lambda_D$ (the Debye length) for waves with $\omega/k \gg v_e$ (electron thermal velocity) and $v_g \gg c_s$ (ion acoustic speed). The last requirement is needed for the assumption of stationary ions. When the phase velocities $\omega/k$ approach the electron thermal velocity, temperature effects can no longer be neglected and the linear dispersion relation must be altered. Further, the effects of trapped particles begin to play an important role[35]. Nonlinear phase shifts of electron plasma waves due to trapped particles have been studied extensively both theoretically and experimentally, recently by Sugai and Märk[36], who give references to related work. It is interesting to note that the nonlinear frequency shift introduced by the trapped particles is proportional to the square root of the wave amplitude[37], while the frequency shift deduced in Sec. VI is proportional to the square of the wave amplitude. The modulational instability of an electron plasma resulting from the trapped electrons was investigated theoretically by Dewar et al.[38] for an infinite plasma. These authors used a nonlinear Schrödinger equation similar to Eq. 62 with Q replaced by the frequency shift resulting from the trapped particles. They found that trapped particles could give rise to modulational instability, at least as long as the fraction of particles was sufficiently small.

In conclusion we consider a numerical example of the modulational instability in a typical Q-machine plasma with $n = 5.10^8$ cm$^{-3}$, $T_e = T_i = 0{,}2$ eV and $r_o/\lambda_D = 100$. Electron plasma waves in such a plasma have, for $\omega/k \gg v_e$, been found to follow approximately the dispersion relation Eq. (15)[39]. Generally, in plasma wave experiments the wave is launched at one end of the machine and detected at the other end. Thus, in order to observe the modulational instability, we must have a spatial growth length of the same order of size as the machine length, typically 1 m. The spatial growth rate $K_i$ can be obtained from Eq. (76) by using the fact that the modulation in the $(x,t)$ space approximately propagates with the group velocity $v_g$, thus we have

$$K_i = \frac{\gamma_m}{v_g} = \frac{Q}{v_g}\rho_o^2 \tag{82}$$

(note that $K_i$ is normalized with respect to $\beta$). Now let us consider an electron plasma wave with wavenumber $k = 3/\beta$. This wave will easily fulfil the conditions for applying the theory, $\omega/k \simeq 13\ v_e$ and $v_g \simeq 0.5\ (M/m)^{\frac{1}{2}}\ c_s$ (M and m are the mass of ions and electrons, respectively, $(M/m)^{\frac{1}{2}} \simeq 500$ for Cs-plasma). In order that the spatial growth length $L = (2\pi/K_i)\beta$ equals 1 m, Eq. (82) yields $\rho_o = 0.14$. From this, noting that $\rho_o$ is the unperturbed electric field normalized with respect to $E_o$ $(= m\omega_p^2\ \beta/e)$, we find the wave potential $\phi_o$ needed to give a growth length of the order of the machine length:

$$\frac{e\phi_o}{m u_o^2} = \frac{\rho_o}{k\beta} = 0.05$$

$(u_o = \beta\omega_p)$. With reference to the work of Saeki[12)], it seems possible to obtain such a wave amplitude in a Q-machine plasma. However, since other nonlinear effects will compete with the evolution of the modulational instability as discussed above, one cannot conclusively predict that the instability will be detectable in a Q-machine plasma. Therefore it seems reasonable to consider the possibility of an experimental verification.

## VIII. Acknowledgement

The author would like to thank P. Michelsen and H.L.Pécseli for valuable discussions.

This work was performed under the contract of association between Research Establishment Risö and Euratom.

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**APPENDIX**

$$D_{13} = 3\,\frac{k^3(2\omega^2+1)}{\omega^7}\,\alpha^2 a^3$$

$$D_{12} = \left(3\,\frac{2k^2-1}{k\omega}B_1 + 4i\frac{k^2}{\omega^3}b\right)\alpha a$$

$$D_{11} = -2\,\frac{\omega^3}{k^5}\left(B_1\frac{\partial B_1}{\partial a} + \bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right) + i\omega\left(B_1\frac{\partial b}{\partial a} + \bar{B}_1\frac{\partial b}{\partial \bar{a}}\right)$$
$$- \frac{k^3(2\omega^2+1)}{\omega^7}\alpha^2 a^2 \bar{a} - \frac{k}{\omega}\left(c_2 + \frac{k}{\omega}c_1\right)\alpha a + i\left(A_2\frac{k}{\omega^2} + \frac{1}{\omega}B_2\right)$$

$$D_{10} = B_1\left(-2\frac{k}{\omega^3}\alpha\bar{a} + \frac{\omega^3}{k^3}\,\frac{\partial c_2}{\partial a} - \frac{\partial c_1}{\partial a}\right)$$

$$D_{23} = \frac{3k^2(1+\omega^2)}{2\omega^6}\alpha^2 a^3$$

$$D_{22} = \left(\frac{k^4-3k^2-1}{k^4}B_1 + 2\,\frac{ik}{\omega^2}b\right)\alpha a$$

$$D_{21} = \frac{\omega^4}{k^6}\left(B_1\frac{\partial B_1}{\partial a} + \bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right) - i\,\frac{\omega^2}{k^3}\left(B_1\,\frac{\partial b}{\partial a} + \bar{B}_1\frac{\partial b}{\partial \bar{a}}\right)$$
$$- \frac{k^2(1+\omega^2)}{2\omega^6}\alpha^2 a^2\bar{a} - \frac{k}{\omega}c_1\alpha a + \frac{i}{\omega}A_2$$

$$D_{20} = B_1\left(-\frac{1}{\omega^2}\alpha\bar{a} + \frac{\omega^3}{k^3}\frac{\partial c_1}{\partial a}\right)$$

$$D_{32} = \frac{k^2(1-5\omega^2)}{\omega^6}B_1\alpha a$$

$$D_{31} = -\frac{1}{k^2}\left(B_1\frac{\partial B_1}{\partial a} + \bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right) + \frac{ik(1-2\omega^2)}{\omega^2}\left(B_1\frac{\partial b}{\partial a} + \bar{B}_1\frac{\partial b}{\partial \bar{a}} + B_2\right)$$

$$D_{30} = -B_1\frac{\partial c_2}{\partial a}$$

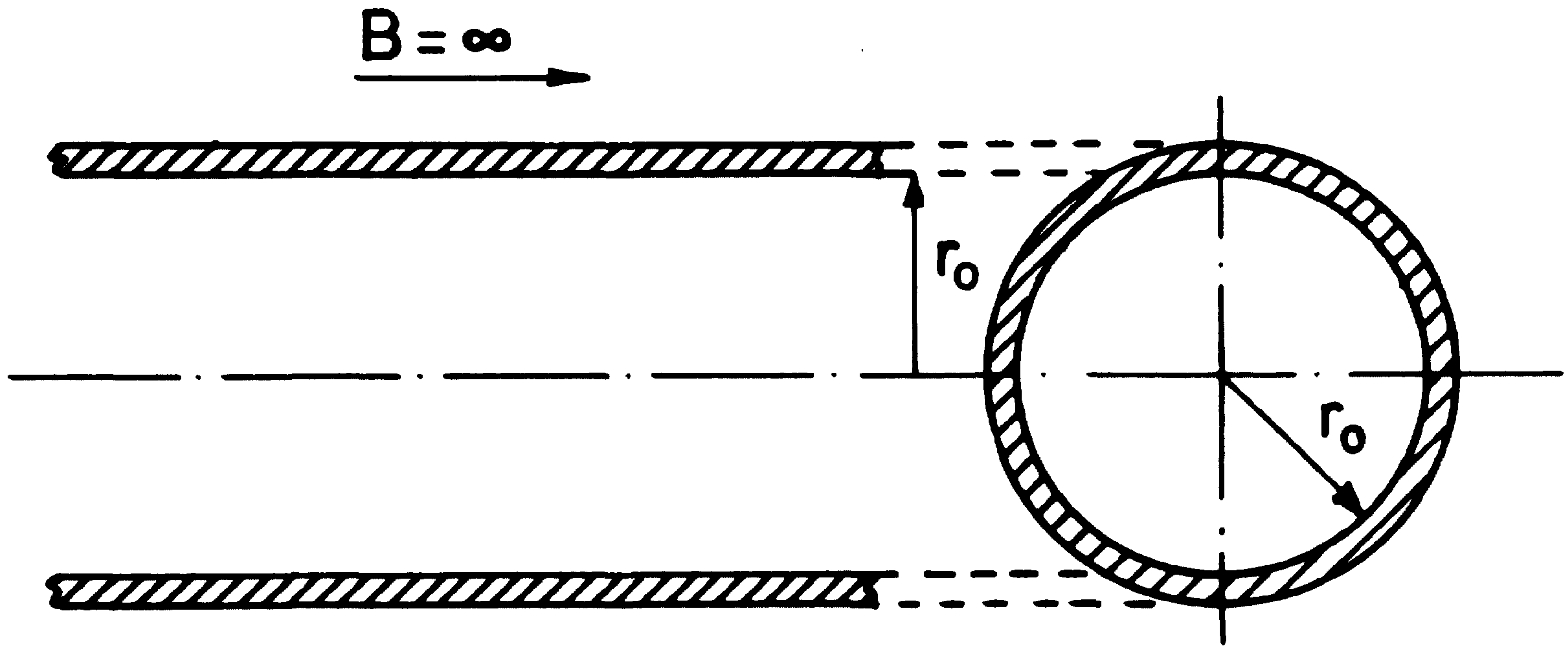


Fig. 1. Schematic of the plasma-filled waveguide.

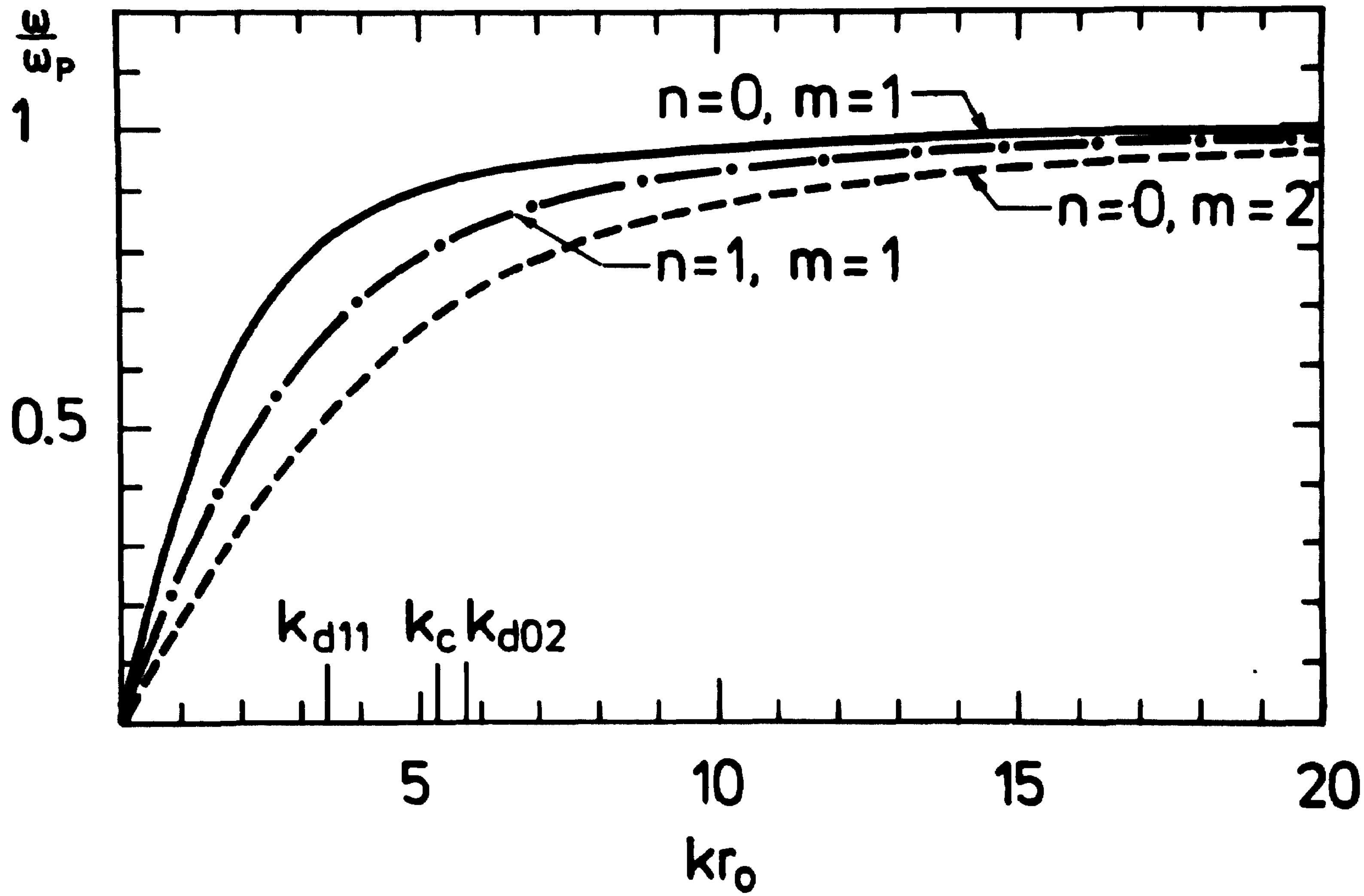


Fig. 2. Dispersion curves for the lowest order modes. $k_{dnm}$ indicates the lower limit for decay of the (n=o, m=1) mode into the (n, m) mode. $k_c$ is the critical wavenumber for modulational instability, waves with $k > k_c$ are unstable.

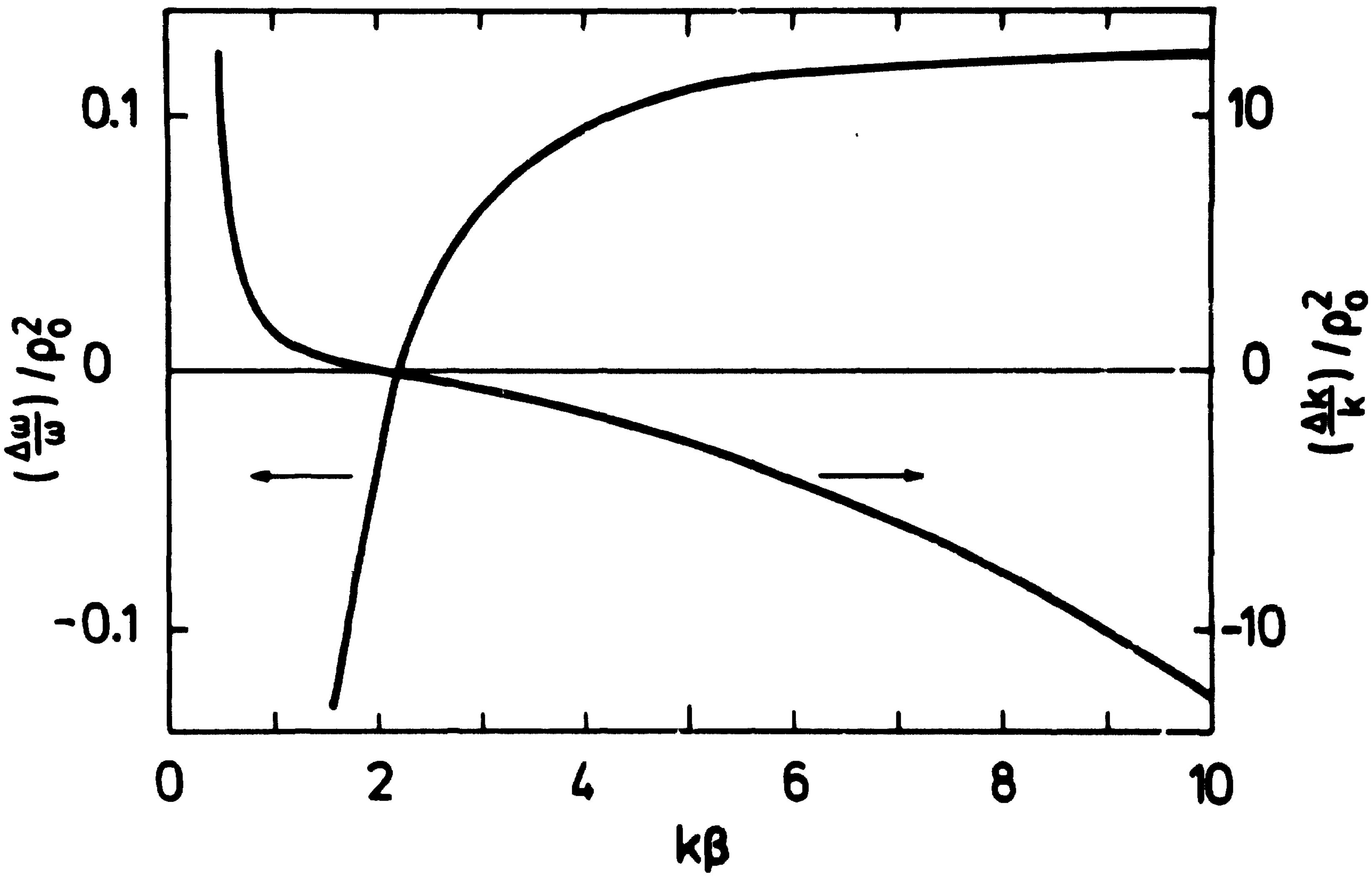


Fig. 3. The amplitude dependent frequency ($\Delta\omega/\omega$) and wavenumber ($\Delta k/k$) shifts versus the wavenumber of the electron plasma wave. $\rho_o$ is the normalized wave amplitude, $\beta = r_o/2.405$.

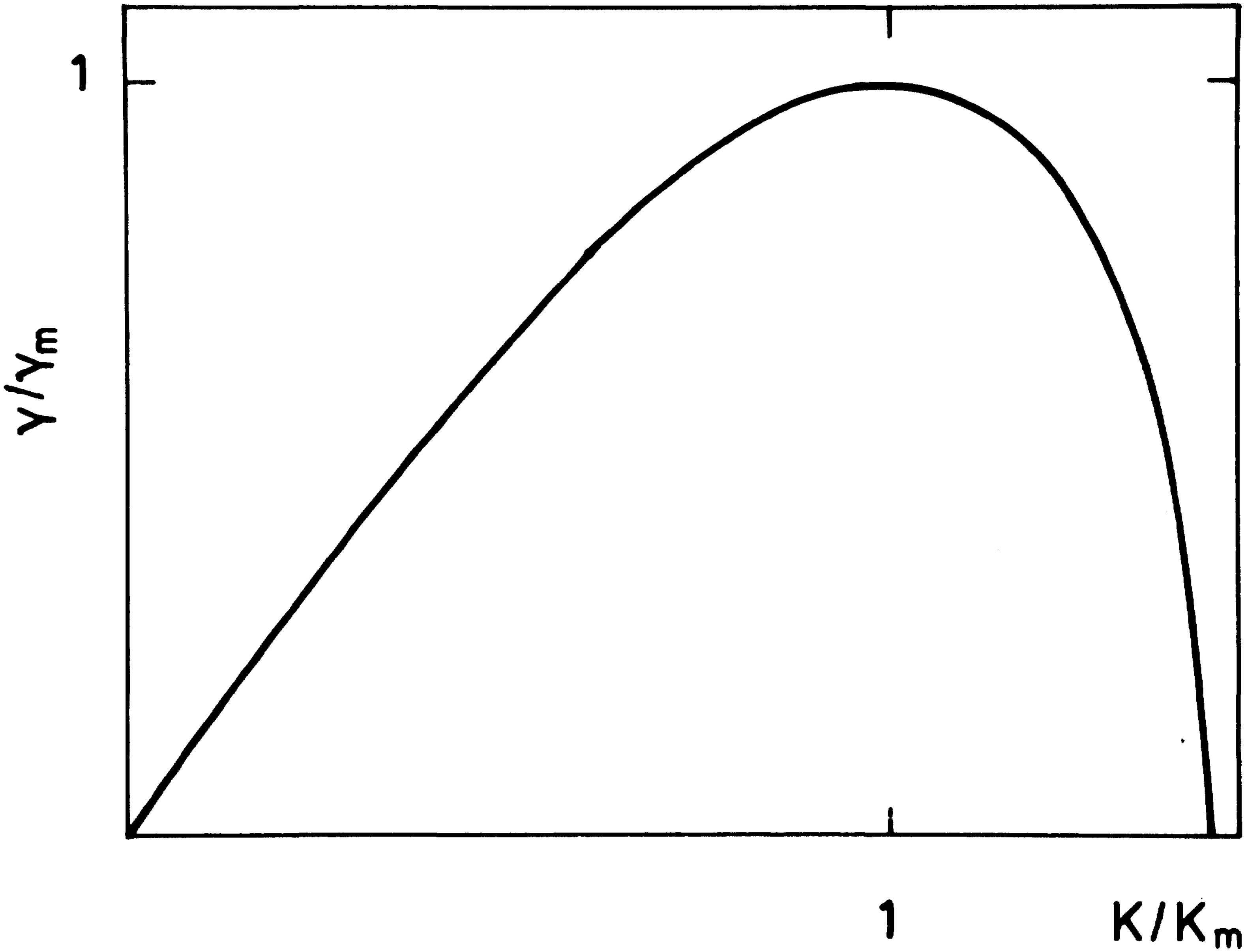


Fig. 4. The growth rate of the modulational instability, γ, , versus its wavenumber, K .

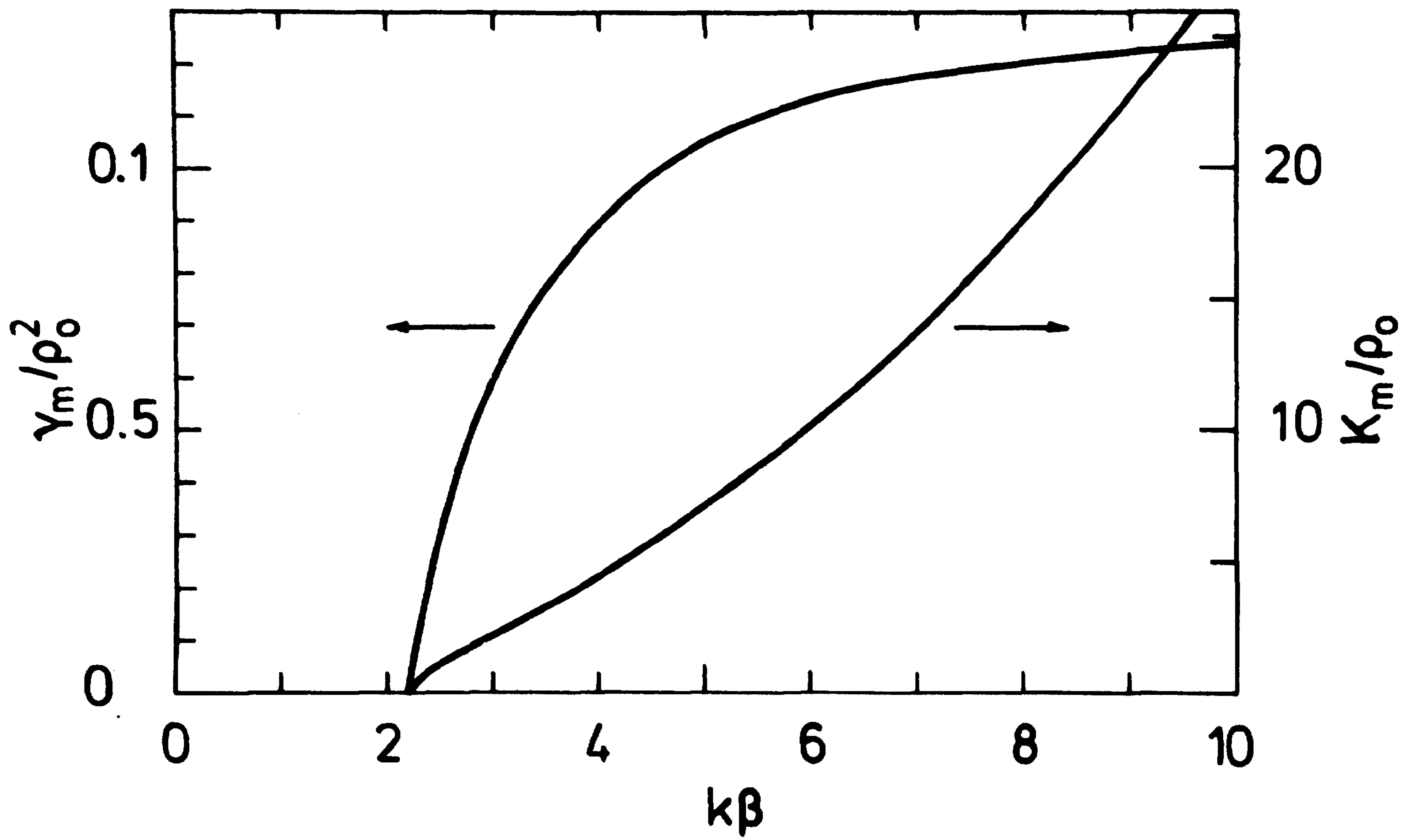


Fig. 5. The maximum growth rate, $\gamma_m$, and the corresponding wavenumber, $K_m$, of the modulation versus the wavenumber of the electron plasma wave. $\rho_o$ is the normalized wave amplitude, $\beta = r_o/2.405$.

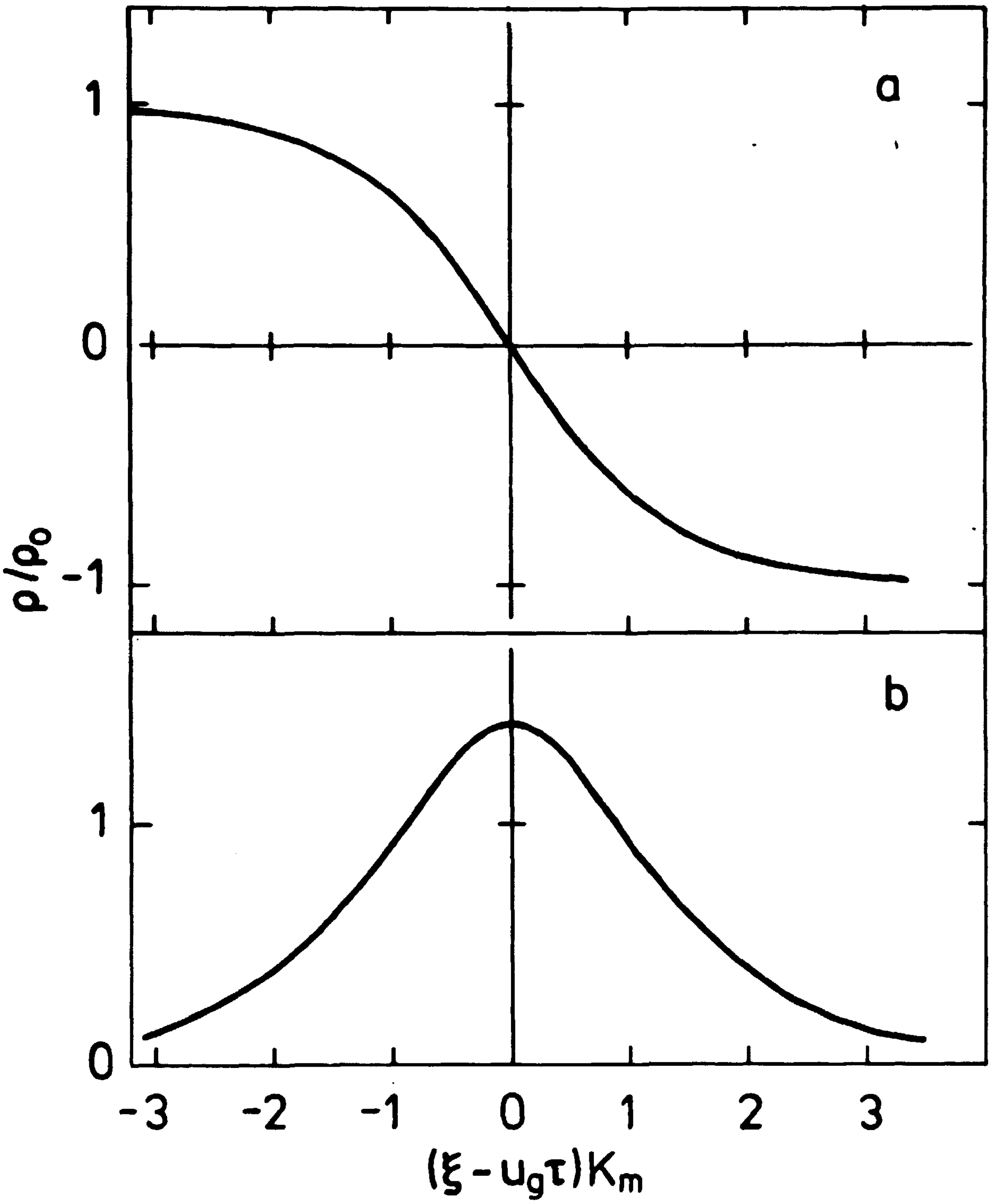


Fig. 6. Equilibrium solutions to the nonlinear Schrödinger equation.
a) The "envelope shock",
b) The "envelope soliton".